2021

MATHEMATICS — HONOURS

Paper : DSE-A-1

(Bio-Mathematics)

Full Marks : 65

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

Group – A

(Marks : 20)

1. Answer the following multiple choice questions with only one correct option. Choose the correct option with proper justification. (1+1)×10

(a) The equilibrium point (3, 0) of the following two-dimensional model $\frac{dx}{dt} = x\left(1 - \frac{x}{3}\right) - xy,\ \frac{dy}{dt} = (x-1)y$ is a

(i) stable node (ii) unstable saddle

(iii) locally asymptotically stable (iv) none of these.

(b) The Holling type-III functional response $\varphi(N)$ represents in $(N, \varphi(N))$ plane

(i) a sigmoidal curve (ii) a closed curve

(iii) a hyperbolic curve (iv) a straight line.

(c) In Gompertz growth model $\frac{dP}{dt} = CP\, ln(K/P)$, the population ($P$) grows fastest when P is equal to

(i) 0 (ii) K

(iii) e/K (iv) K/e

C, K being positive parameters.

(d) What type of bifurcation will occur in the system $\frac{dx}{dt} = \mu x - x^3$, where μ is bifurcation parameter?

(i) Saddle-node bifurcation (ii) Pitchfork bifurcation

(iii) Transcritical bifurcation (iv) None.

(e) A two-dimensional system has characteristic equation $\lambda^2 + \alpha\lambda + \alpha\beta(1-\alpha) = 0$, (where $\alpha > 0,\ \beta > 0$) at the equilibrium $(\alpha, 1-\alpha)$. If $\frac{4\beta}{1+4\beta} < \alpha < 1$, then the equilibrium is a

(i) unstable focus (ii) unstable node

(iii) stable focus (iv) stable node.

(f) In the two-dimensional system

$$\frac{dx}{dt} = 2 - x + \frac{x^2}{y},$$
$$\frac{dy}{dt} = x^3 - y$$

(i) x is activator of y and y is inhibitor of x

(ii) y is activator of x and x is inhibitor of y

(iii) both of them are activator and inhibitor

(iv) none of these.

(g) The equilibrium point $x = k$ for the equation $\frac{dx}{dt} = rx\left(1 - \left(\frac{x}{k}\right)^{\theta}\right)$, where r, k, θ are positive parameters, is

(i) unstable (ii) stable

(iii) stable but not asymptotically stable (iv) none of these.

(h) Consider a dynamical system $\frac{dr}{dt} = r(1-r)(r-2)(r-3)$, $\frac{d\theta}{dt} = 1$, where (r, θ) be the polar coordinates on the plane. The number of limit cycles is

(i) 1 (ii) 2

(iii) 3 (iv) 4.

(i) The fixed point $x^* = \frac{\alpha - 1}{\beta}$ of the difference equation $x_{n+1} = \frac{\alpha x_n}{1 + \beta x_n}$, $\alpha > 1,\ \beta > 0$ is

(i) unstable (ii) asymptotically stable

(iii) stable but not asymptotically stable (iv) none of these.

(j) The system

$$\frac{dx}{dt} = -y + x\left(x^2 + y^2 - 1\right)$$
$$\frac{dy}{dt} = x + y\left(x^2 + y^2 - 1\right)$$

has no closed orbit inside the circle

(i) $x^2 + y^2 = 1$ (ii) $x^2 + y^2 = 2$

(iii) $x^2 + y^2 = \frac{1}{2}$ (iv) $x^2 + y^2 = \frac{1}{4}$.

Group – B

Unit – I

(Marks : 15)

Answer ***any one*** question.

2. (a) State the basic assumptions of spruce budworm population dynamics and construct the model equation with logistic population growth and suitable predation term. Derive the corresponding dimensionless equation.

(b) Consider the following epidemic model :

$$\frac{dS}{dt} = A - rS - \frac{\beta SI}{1+\alpha I},$$
$$\frac{dI}{dt} = \frac{\beta SI}{1+\alpha I} - \mu I,$$

where A, r, α, β, μ are positive parameters. Find the equilibrium points of the system and discuss the nature of the equilibrium points.

(c) Write short notes on the following :

(i) Gompertz growth

(ii) Basic reproduction number. (3+2)+(3+3)+(2+2)

3. (a) Consider the growth model $\frac{dN}{dt} = rN\left(\frac{N}{A} - 1\right)\left(1 - \frac{N}{K}\right)$, where r, A, K are positive parameters and $A < K$. Determine all the equilibrium points and discuss their stability.

(b) Consider the following harvesting model :

$$\frac{dN}{dt} = rN(1 - N/K) - qEN,$$

where E is the fishing effort, q is the catchability rate, N is the stock level, r is the growth rate, K is the carrying capacity. Investigate the stability of the equilibrium points and show that the maximum sustainable yield is $rK/4$.

(c) Consider the following competitive model :

$$\frac{dx}{dt} = x(1 - x - \alpha y),$$
$$\frac{dy}{dt} = \rho y(1 - y - \beta x),$$

where α, β, ρ are positive constants. Show that if $\alpha > 1$ and $\beta < 1$, the first species is going to extinction and second species will be surviving with its carrying capacity but the opposite phenomenon occur when $\alpha < 1$ and $\beta > 1$. (3+2)+(3+2)+5

Unit – II

(Marks : 20)

Answer ***any two*** questions.

4. (a) Let (x^*, y^*) be an equilibrium point of the following system :

$$\frac{dx}{dt} = f(x, y), \; \frac{dy}{dt} = g(x, y),$$

where f and g are continuously differentiable.

(i) Obtain the linearized system about (x^*, y^*).

(ii) Hence discuss stability of (x^*, y^*).

(b) State the basic assumptions of classical Lotka-Volterra model for a predator-prey system. Write the model equations. Discuss the stability of the system about the non-trivial equilibrium. (2+3)+(2+1+2)

5. (a) Consider the nonlinear system :

$$\frac{dx}{dt} = x\left\{2\left(1 - \frac{x}{k}\right) - \frac{3y}{1+x}\right\},$$
$$\frac{dy}{dt} = y\left(-\frac{1}{2} + \frac{x}{1+x}\right), \; k > 0.$$

Find the equilibrium points and discuss their stability nature.

(b) What is meant by bifurcation? Discuss the saddle-node bifurcation for the system $\frac{dx}{dt} = \mu - x^2$, μ is the parameter. (2+4)+(1+3)

6. Consider the model following model of bacterial growth in a chemostat :

$$\frac{dN}{dt} = \left(\frac{k_1 C}{k_2 + C}\right) N - \frac{FN}{V},$$

$$\frac{dC}{dt} = -\alpha\left(\frac{k_1 C}{k_2 + C}\right) N - \frac{FC}{V} + \frac{FC_0}{V},$$

where the symbols have their usual meanings.

(a) Show that the equations can be reduced to the following dimensionless form by the substitution $N = \frac{Fk_2}{\alpha V k_1} u, C = k_2 \nu, \ t = \frac{V}{F}\tau$:

$$\frac{du}{d\tau} = \alpha_1\left(\frac{\nu}{1+\nu}\right) u - u,$$

$$\frac{dv}{d\tau} = -\left(\frac{\nu}{1+\nu}\right) u - \nu + \alpha_2,$$

where α_1 and α_2 are the parameters to be determined by you.

(b) Find the equilibrium points of the dimensionless system. Find the conditions on α_1 and α_2 so that the equilibrium points become biologically meaningful.

(c) Determine the stability of the biologically meaningful equilibrium points. 2+3+5

7. What is a compartmental model? State the basic assumptions of Kermack-McKendrick SIR compartmental model. Draw the flowchart and write the model equations. Find the basic reproduction number. Determine the conditions for which the epidemic spreads and infection dies out. 2+2+1+2+3

Unit – III

(Marks : 10)

Answer ***any one*** question.

8. (a) Suppose x^* is a fixed point of the system $x_{n+1} = f(x_n)$, where $f(x)$ is a continuously differentiable function and $|f'(x^*)| \neq 1$. Prove that x^* is asymptotically stable if $|f'(x^*)| < 1$ and unstable if $|f'(x^*)| > 1$.

(b) Consider the following non-linear difference equation

$$x_{n+1} = \frac{\lambda x_n}{\mu + x_n}, \text{ where } \lambda > 0, \ \mu > 0.$$

Find the fixed points and discuss their stability. 4+(3+3)

Please Turn Over

9. (a) Solve the following non-homogeneous system and discuss the stability of the fixed point by using Cobweb diagram :

$$x_{n+1} = \frac{3}{4} x_n + 10.$$

(b) Consider the discrete-time predator-prey system :

$$x_{n+1} = a x_n (1 - x_n) - b x_n y_n,$$
$$y_{n+1} = -\, c y_n + d x_n y_n,$$

where a, b, c, d are positive parameters. Find the fixed points of the system and discuss their stability. (2+2)+(3+3)