2021

MATHEMATICS — HONOURS

Paper : CC-3

(Real Analysis)

Full Marks : 65

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

$\mathbb{N}$, $\mathbb{R}$, $\mathbb{Q}$ denote the set of all natural, real and rational numbers respectively.

Notations and symbols have their usual meanings.

1. Answer *all* the following multiple choice questions. For each question **1** mark for choosing correct option and **1** mark for justification. 2×10

(a) Let $A=[0,1]$ and $B=\left\{\frac{1}{n}: n\in\mathbb{N}\right\}$. Then $A-B$ is

(i) an open set (ii) a closed set

(iii) neither an open nor a closed set (iv) a clopen set.

(b) Let $B=\left\{\frac{1}{2n}+\frac{1}{3m}: n,m\in\mathbb{N}\right\}$. Then $\frac{1}{3}$ is

(i) a limit point of the set but not an element of the set

(ii) not a limit point of the set but an element of the set

(iii) both an element and a limit point of the set

(iv) neither an element nor a limit point of the set.

(c) If A, B are bounded subsets of $\mathbb{R}$ such that Sup $A <$ Inf B, then

(i) $A \subset B$ (ii) $A \cap B = \phi$

(iii) $A \cap B \neq \phi$ (iv) $A = B$.

(d) The set $T=\{(x_1, x_2,..., x_n) : x_1, x_2,..., x_n \in \{1, 3, 5, 7, 9\}\}$ is

(i) empty (ii) finite

(iii) enumerable (iv) uncountable.

(e) Which of the following statement is true?

(i) Every bounded set has a limit point

(ii) Every infinite set has a limit point

(iii) Every bounded infinite set has an interior point

(iv) Every uncountable set has a limit point.

(f) If $0 < x < 2021$, then $\lim_{n\to\infty}\left(\frac{x^{n+1}+2021^{n+1}}{x^n+2021^n}\right)$ is

(i) 0 (ii) x

(iii) 2021 (iv) 1.

(g) If $x_n = \frac{1+\sin\left(n^2+1\right)\pi+\cos\left(n^3-5\right)\pi}{n+1}$, then $\limsup x_n$ is

(i) 3 (ii) 2

(iii) 1 (iv) 0.

(h) Identify the incorrect statement from the following :

(i) Every convergent sequence is bounded.

(ii) Every bounded sequence has at least one subsequential limit.

(iii) Unbounded sequence cannot have any subsequential limit.

(iv) Unbounded sequence is never a Cauchy sequence.

(i) Let $\{a_n\}$ be a sequence of positive integers and bounded by a positive integer k. Then the series $\sum_{n=1}^{\infty}\frac{a_n}{(k+1)^n}$

(i) must be convergent (ii) must be divergent

(iii) may be convergent (iv) may be divergent.

(j) The series $\sum_{n=1}^{\infty}\frac{2n^2}{1+n^2}$

(i) converges to 0 (ii) converges to 1

(iii) diverges (iv) converges to 2.

Unit - 1

Answer ***any four*** questions.

2. (a) Let, S, T be two non-empty bounded below sets of real numbers such that $S \subseteq T$. Prove that $Inf\, S \geq Inf\, T$.

(b) Let A and B be two non-empty bounded subsets of $\mathbb{R}$.

Prove that $inf(A \cup B) = \text{Minimum } \{inf\, A,\ inf\, B\}$. 3+2

3. Show that $\mathbb{N} \times \mathbb{N}$ is enumerable. Hence prove that $S = \{3^i 5^j : i, j \in \mathbb{N}\}$ is enumerable. 3+2

4. (a) Prove or disprove : The complement of any non-empty finite set in $\mathbb{R}$ has at least one limit point in $\mathbb{R}$.

(b) Find the derived set of $\left\{x \in \mathbb{R} : x^2 - 4x + 3 > 0\right\}$. 3+2

5. (a) Give an example to show that an infinite intersection of open sets is the set $\left\{\sin\frac{\pi}{7}\right\}$.

(b) Prove or disprove : Bolzano-Weierstrass theorem cannot be verified with this set $\left\{n^{(-1)^n} : n \in \mathbb{N}\right\}$.

(c) Prove or disprove : The set of all irrational numbers in $\left[\sqrt{2}, \sqrt{3}\right]$ is uncountable. 2+2+1

6. Prove that no non-empty proper subset of $\mathbb{R}$ is both open and closed. 5

7. (a) Prove or disprove : If A is a non-empty open set of real numbers, then A is uncountable.

(b) Prove or disprove : $[0, 3] - \bigcup_{n=1}^{\infty}\left(1 - \frac{1}{n}, 1 + \frac{1}{n}\right)$ is a closed set. 3+2

8. (a) If $a > 0$, show that the set $\{aq : q \in \mathbb{Q}\}$ is dense in $\mathbb{R}$.

(b) Check whether the set $\left\{\frac{1}{n} : n \in \mathbb{N}\right\} \cup \left\{-\frac{1}{n} : n \in \mathbb{N}\right\} \cup \{0\}$ is a neighbourhood of '0' or not. 3+2

Unit - 2

Answer ***any four*** questions.

9. (a) Prove or disprove : If $\lim x_n = 0$, then $\lim \frac{y_n}{n} = 0$, where $y_n = \sum_{i=1}^{n} x_i$.

(b) Prove that the sequence $\{x_n\}$ is bounded, where $x_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{n!}$. 3+2

Please Turn Over

10. (a) Prove or disprove : If $\{x_n\}$ and $\{y_n\}$ are sequences of real numbers such that $\{x_n y_n\}$ is convergent, then both $\{x_n\}$ and $\{y_n\}$ are convergent.

(b) Prove that the sequence $\left\{\sqrt[5]{n^9}\right\}$ diverges to $+\infty$. 3+2

11. Prove that every Cauchy sequence is convergent. Examine whether $\{x_n\}$ is a Cauchy sequence where $x_n = \frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2n} \forall n \in \mathbb{N}$. 3+2

12. Prove that every sequence has a monotone subsequence. 5

13. (a) State Sandwich theorem for three sequences of real numbers. Using it find $\lim\limits_{n\to\infty} \frac{\cos\left(3n^3+4n^2-7\right)}{n^4+1}$.

(b) Prove or disprove : If $\{x_n\}$ is a bounded sequence of real numbers and $\lim y_n = 0$, then $\lim x_n y_n = 0$. (1+2)+2

14. (a) Find the upper and lower limits, if exist, of $\left\{\frac{2x^2}{7}-\left[\frac{2x^2}{7}\right]\right\}$, where $[x]$ is the greatest integer $\leq x$.

Is the sequence convergent? Justify.

(b) Prove or disprove : For any two bounded sequences $\{x_n\}$ and $\{y_n\}$ of real numbers, $\underline{\lim}(x_n+y_n) = \underline{\lim}\, x_n + \overline{\lim}\, y_n$. (2+1)+2

15. (a) Prove or disprove : If $\{I_n\}$ is a sequence of non-empty open intervals such that $I_{n+1} \subset I_n$ and $\lim l(I_n) = 0$, where $l(I_n)$ stands for length of I_n, then $\bigcap\limits_{n\in\mathbb{N}} I_n$ contains exactly one real number.

(b) Show that $\lim\limits_{n\to\infty} \frac{(n!)^{\frac{1}{n}}}{n} = \frac{1}{e}$. 3+2

Unit - 3

Answer ***any one*** question.

16. (a) Test the convergence of the infinite series $\sum\limits_{n=1}^{\infty}(-1)^n \frac{n+2}{2^n+7}$.

(b) Prove or disprove : If $\sum u_n$ with $u_n > 0$ is convergent, then $\sum\sqrt{u_n u_{n+1}}$ is convergent. 3+2

17. State Gauss test. Test the convergence of the series :

$$\left(\frac{1}{2}\right)^2+\left(\frac{1.3}{2.4}\right)^2+\left(\frac{1.3.5}{2.4.6}\right)^2+\ldots$$ 1+4