2021

MATHEMATICS — HONOURS

Seventh Paper

(Module : XIII)

Full Marks : 50

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

$\mathbb{R}$, $\mathbb{C}$ respectively denote the set of all real numbers and complex numbers

Group - A

[Analysis - IV]

(Marks : 20)

Answer ***any two*** questions.

1. (a) Let f and g be positive-valued functions defined on $[a, b]$ such that both f and g have infinite discontinuities only at 'a', both are bounded and R-integrable on $[a+\varepsilon, b]$ for $0 < \varepsilon < b-a$.

If $\underset{x\to a}{\text{Lt}} \dfrac{f(x)}{g(x)} = l$, where l is a non-zero real number, prove that $\int_a^b f(x)dx$ and $\int_a^b g(x)dx$ converge or diverge together.

(b) Test the convergence of $\int_0^1 x^{n-1} \log x\, dx$. 5+5

2. (a) Examine the convergence of $\int_0^\pi \frac{\sin x}{x^p} dx$.

(b) Show that the integral $\int_0^\infty e^{-ax} \frac{\sin x}{x} dx$ is convergent if $a \geq 0$.

(c) Examine the convergence of $\int_0^1 x^{m-1} \left(\log_e x\right)^{n-1} dx$. 3+3+4

3. (a) Obtain the Fourier series expansion of $f(x)$ in $[-\pi, \pi]$ where

$$f(x)=\begin{cases} 0, & -\pi \leq x < 0 \\ \frac{1}{4}\pi x, & 0 \leq x \leq \pi \end{cases}$$

Hence show that the sum of the series $1+\frac{2}{3^2}+\frac{2}{5^2}+\ldots.$ is $\frac{\pi^2}{8}$.

(b) Evaluate $\int_0^1 dy \int_y^1 e^{x^2}\, dx$. 7+3

4. (a) Let a function f be defined on a rectangle $R = [0, 1 ; 0, 1]$ as follows :

$$f(x,y)=\begin{cases} \frac{1}{2}, & \text{when } y \text{ is rational} \\ x, & \text{when } y \text{ is irrational} \end{cases}$$

Show that

(i) $\int_0^1 dx \int_0^1 f(x,y)dy$ does not exist, but

(ii) $\int_0^1 dy \int_0^1 f(x,y)dx = \frac{1}{2}$.

(b) Evaluate $\iint_R \sqrt{\left|x^2-2y\right|}\, dx\, dy$ where $R = [-2, 2 ; 0, 2]$

Or,

Show that $\iint_E \frac{dx\, dy}{\sqrt{(x+y+1)^2-4xy}} = \frac{1}{2}\log_e\left(\frac{16}{e}\right)$

by using the transformation $x = u(1+v)$, $y = v(1+u)$, where E is the triangle with vertices (0, 0), (2, 0), (2, 2). (3+2)+5

Group - B

[Metric Space]

(Marks : 15)

5. Answer ***any three*** questions :

(a) (i) Let X be the set of all sequences of real numbers and let $x = \{x_n\}_n$ and $y = \{y_n\}_n$ be any two members of X. Define $d : X \times X \to \mathbb{R}$ by,

$$d(x, y) = \sum_{n=1}^{\infty} \frac{1}{5^n} \frac{|x_n - y_n|}{1 + |x_n - y_n|}$$

Show that d is a metric on X.

(ii) Show that in a trivial metric space (X, d), every subset A of X is both open and closed. 3+2

(b) (i) Prove that in a metric space (X, d) every open ball is an open set. Is the converse true? Justify your answer.

(ii) Let X denote the set of all Riemann integrable functions on $[a, b]$. For $f, g \in X$, define $d(f, g) = \int_a^b |f(x) - g(x)|\, dx$. Is d a metric on X? Justify. (2+1)+2

(c) (i) Let (X, d) be a metric space and let A, B be two subsets of X. Then show that

$$\text{diam}\,(A \cup B) \le \text{diam}\,(A) + \text{diam}\,(B) + d(A, B),$$

where $d(A, B)$ denotes the distance between two sets A and B.

(ii) Let $A = \{(x, y) : x^2 + y^2 = 2\}$ and $B = \{(x, y) : (x - 1)^2 + y^2 = 2\}$.

Find the diameter of the sets $A \cup B$ and $A \cap B$ with respect to the usual metric on $\mathbb{R}^2$. 3+2

(d) (i) Let $\{x_n\}$ be a sequence in a complete metric space (X, d) with the property that $\sum_{k=1}^{\infty} d(x_k, x_{k+1}) < \infty$. Show that $\{x_n\}$ is convergent.

(ii) Let $\{x_n\}_n, \{y_n\}_n$ be sequences in a metric space (X, d) such that $x_n \to x$ in X. Then $y_n \to x$ in X if and only if $d(x_n, y_n) \to 0$ in $\mathbb{R}$. 3+2

(e) Let (x, d) be a complete metric space and $\{F_n\}_n$ be a descending sequence of non-empty closed sets in X with diameter of F_n tending to 0 as $n \to \infty$. Prove that $F = \bigcap_{n=1}^{\infty} F_n$ contains exactly one point. 5

Group - C

[Complex Analysis]

(Marks : 15)

6. Answer ***any three*** questions :

(a) (i) Define stereographic projection. Find the image point on the Riemann sphere $x^2 + y^2 + \left(z - \frac{1}{2}\right)^2 = \frac{1}{4}$ for the point $4 + 3i$ in the complex plane.

(ii) Show that if f is analytic in a domain $D(\subset \mathbb{C})$ and $|f(z)|$ is constant in D, then the function $f(z)$ is constant in D. 2+3

(b) Let G be region in $\mathbb{C}$ and $(x_0, y_0) \in G$. If $u, v : G \to \mathbb{R}$ be differentiable at (x_0, y_0), prove that, $u + iv : G \to \mathbb{C}$ is differentiable at (x_0, y_0), provided u, v satisfy the Cauchy–Riemann equations at (x_0, y_0). 5

(c) Examine the continuity and differentiability at (0, 0) and the Cauchy–Riemann equations at (0, 0) for the following function defined by

$$f(z) = \begin{cases} \dfrac{x^5 - y^5}{x^2 + y^2} + i\dfrac{x^5 + y^5}{x^2 + y^2}, & \text{if } (x, y) \neq (0, 0) \\ 0, & \text{if } (x, y) = (0, 0) \end{cases}$$ 5

(d) (i) Let $u = x^2 - y^2, v = \dfrac{-x}{x^2 + y^2}$. Prove that both u and v satisfy Laplace's equation but $u + iv$ is not an analytic function on the complex plane.

(ii) If $f(z)$ and $g(z)$ be both analytic in a domain $D(\subset \mathbb{C})$ and if $f(z)\, g(z) = U(x, y) + iV(x, y)$, then show that U and V are both harmonic in D. 2+3

(e) Define a harmonic function in a region $G(\subseteq \mathbb{C})$. Prove that $u(x, y) = x^3 - 3xy^2$ is a harmonic function on $\mathbb{C}$. Determine its conjugate harmonic and corresponding analytic function.

1+2+2