

2021

MATHEMATICS — HONOURS

Paper : DSE-A(2)-2

(Mathematical Modelling)

Full Marks : 65

*The figures in the margin indicate full marks.**Candidates are required to give their answers in their own words as far as practicable.*

Group - A

1. Answer *all* questions :

2×10

(To each question there are four alternatives of which one is correct. Choose the correct one with proper justification. One mark is for correct answer and one mark is for proper justification.)

(a) Laplace transform of a function $f(t)$ is given by

(i) $F(s) = \int_0^{\infty} f(t)e^{st} dt$, where $s > 0$

(ii) $F(s) = \int_0^{\infty} f(t)e^{-st} dt$, where $s > 0$

(iii) $F(s) = \int_0^{\infty} f(s)e^{-st} dt$, where $s > 0$

(iv) $F(s) = \int_0^{\infty} f(-s)e^{-st} dt$, where $s > 0$.

(b) The point $x = 0$, in the standard Bessel's equation is a (an)

(i) ordinary point

(ii) regular singular point

(iii) irregular singular point

(iv) none of these.

(c) Legendre polynomial $P_1(x)$ is given by

(i) 1

(ii) $x/2$ (iii) x (iv) x^2 .(d) Using linear congruence method $x_{n+1} = 7x_n + 13 \pmod{97}$ with seed $x_0 = 105$, three random numbers generated are respectively

(i) 63, 13 and 90

(ii) 13, 34 and 78

(iii) 69, 11 and 90

(iv) none of these.

(e) Number of extreme points of the convex set $x^2 + y^2 \leq 1$ are

(i) two

(ii) three

(iii) countably infinite

(iv) uncountably infinite.

Please Turn Over

- (f) If $P_n(x)$ is Legendre polynomial of degree n , then
- (i) $P_n(-x) = P_n(x)$
 - (ii) $P_n(-x) = (-1)^n P_n(x)$
 - (iii) $P_n(-1) = 1$
 - (iv) none of these.
- (g) Cars arrive at an automated car wash following Poisson distribution. If their arrival rate is 20 per hour, and it takes exactly 2 minutes for a car wash, what is the average waiting time in line?
- (i) 1 minute
 - (ii) 2 minutes
 - (iii) 4 minutes
 - (iv) 3 minutes.
- (h) The pattern in which the customer change his queue in the initial time and finally join the shortest queue is
- (i) Balking
 - (ii) Reneging
 - (iii) Jockeying
 - (iv) Collusion.
- (i) The set of regular singular points of the ordinary differential equation
- $$x(x^3 + x^2 - x - 1)^2 \frac{d^2y}{dx^2} - (x^4 + 3x^3 + 3x^2 + x) \frac{dy}{dx} + (x^4 - 5x^2 + 4)y = 0$$
- is
- (i) $\{0, -1\}$
 - (ii) $\{0\}$
 - (iii) $\{0, 1, -1\}$
 - (iv) none of these.
- (j) The zeros of Legendre polynomial of 1st kind can be
- (i) any real number
 - (ii) any real or purely imaginary number
 - (iii) any real numbers between -1 and 1
 - (iv) only positive proper fraction.

Group - B

Unit - 1

2. Answer **any two** questions :

(a) (i) $x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + (x^2 - 4^2)y = 0$ near $x = 0$.

(ii) Solve the initial-value problem $\frac{d^2y}{dt^2} - \frac{dy}{dt} - 2y = 18e^{-t} \sin 3t$, $y(0) = 0$, $y'(0) = 3$, using Laplace transform method. 5+5

(b) (i) Prove the Recurrence Formulae for Legendre Polynomials

$$(2n + 1) \times P_n(x) = (n + 1) P_{n+1}(x) + n P_{n-1}(x).$$

(ii) Using generating function for $P_n(x)$, prove the following :

$$P_0(x) = 1, P_1(x) = x, P_2(x) = \frac{1}{2}(3x^2 - 1), P_3(x) = \frac{1}{2}(5x^3 - 3x) \text{ and } P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3).$$

5+5

(c) (i) Show that $L\left\{e^{-2t}(3\cos 6t - 5\sin 6t)\right\} = \frac{3(p-8)}{p^2 + 4p + 40}$.

(ii) Evaluate $L^{-1}\left(\frac{2s+1}{(s-1)^2(s+2)^2}\right)$.

5+5

(d) (i) If $J_n(x)$ is Bessel function of 1st kind, then show that $xJ_n'(x) = nJ_n(x) - xJ_{n+1}(x)$ and hence

$$\text{deduce that } \frac{d}{dx}(x^{-n}J_n(x)) = -x^{-n}J_{n+1}(x).$$

[Here Prime (') denotes the 1st order derivative with respect to x .]

(ii) Using Rodrigue's formula, prove that $P_{n+1}'(x) - P_{n-1}'(x) = (2n+1)P_n(x)$ where $P_n(x)$ is the Legendre polynomial of 1st kind of degree n .

[Here Prime (') denotes the 1st order derivative with respect to x .]

5+5

Unit - 2

3. Answer **any five** questions :

(a) Write a short note on Monte Carlo simulation method to find the area of the bounded region

$$x^2 + y^2 \leq 25, x \geq 0, y \geq 0.$$

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(b) Describe the middle-square method to generate random numbers starting with a four-digit positive integer (seed, say x_0) and state one of its major drawback.

4+1

(c) A small harbour has unloading facility for ships. Only one ship can be unloaded at any one time. Ships arrive for unloading of cargo at the harbour. Time-span between the arrival of successive ships and unloading time of the five ships are given in the table below. Find the average time spent by the ships in the harbour.

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	Ship 1	Ship 2	Ship 3	Ship 4	Ship 5
Time between successive ships (in minutes)	15	35	20	100	35
Unloading time (in minutes)	65	55	60	70	80

(d) A firm makes two products P_1 and P_2 and has production capacity of 18 tonnes per day. P_1 and P_2 require same production capacity. The firm must supply at least 4t of P_1 and 6t of P_2 per day. Each tonne of P_1 and P_2 requires 60 hours of machine work each. Maximum machine hours available are 720. Profit per tonne for P_1 is Rs. 160 and P_2 is Rs. 240. Find optimal solution by graphical method.

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Please Turn Over

(e) Given an average arrival of 20 per hour, is it better for a customer to get a service at a single channel with mean service rate of 22 customers or at one of the two channels in parallel, with the mean service rate of 11 customers for each of the two channel? Assume that both queues are (M/M/s: α /FIFO). 5

(f) Solve the Problem Algebraically to get an Optimal Solution : 5

$$\begin{aligned} &\text{Maximize } 25x_1 + 30x_2 \\ &\text{Subject to} \\ &\quad 20x_1 + 30x_2 \leq 690 \\ &\quad 5x_1 + 4x_2 \leq 120 \\ &\quad x_1, x_2 \geq 0 \end{aligned}$$

(g) Following is the optimal table of the LPP

$$\begin{aligned} &\text{Maximize } Z = -x_1 + 2x_2 - x_3 \\ &\text{Subject to} \quad 3x_1 + x_2 - x_3 \leq 10 \\ &\quad -x_1 + 4x_2 + x_3 \geq 6 \\ &\quad x_2 + x_3 \leq 4 \\ &\text{and } x_j \geq 0, \quad j = 1, 2, 3. \end{aligned}$$

$C_j \rightarrow$			-1	2	-1	0	0	0	-M
C_B	y_B	X_B	y_1	y_2	y_3	y_4	y_5	y_6	y_7
0	y_4	6	3	0	-2	1	0	-1	0
2	y_2	4	0	1	1	0	0	1	0
0	y_5	10	1	0	3	0	1	4	-1
$Z_j - C_j \rightarrow$			1	0	3	0	0	2	M

(M is positive large number)

- (i) Determine the range under which the component C_2 of the profit vector can vary so that the current optimal solution remains optimal.
 - (ii) Determine the range under which the component b_2 of the requirement vector can vary to maintain the optimality of the current optimal solution. 5
- (h) A departmental store has only one cashier. During the rush hours, customers arrive at a rate of 20 customers per hour. The average number of customers that can be handled by the cashier is 24 per hour. Assuming the conditions for use of the single-channel queueing model, find out the followings :
- (i) Probability that cashier is idle
 - (ii) Average number of customers in the system
 - (iii) Average time a customer in the system
 - (iv) Average number of customers in the queue
 - (v) Average time a customer spends in the queue. 1+1+1+1+1