2021

MATHEMATICS — HONOURS

Paper : DSE-A(2)-1

(Differential Geometry)

Full Marks : 65

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

[The symbols used have usual meanings]

1. Answer all the following multiple choice questions. For each question, 1 mark for choosing correct option and 1 mark for correct justification. 2×10

(a) The number of independent components of Christoffel's symbols are

(i) $n(n+1)$ (ii) $n(n-1)$

(iii) $\dfrac{n^2(n+1)}{2}$ (iv) 0.

(b) The value of g^{13} and g^{23} when the metric is given by

$ds^2 = 2\left(dx^1\right)^2 + 3\left(dx^2\right)^2 + 4\left(dx^3\right)^2 - 2dx^2dx^3 - 4dx^1dx^3 - 6dx^1dx^2$ in the Riemannian space V_3 are

(i) $-\dfrac{7}{19}$ and $-\dfrac{4}{19}$ (ii) $-\dfrac{7}{19}$ and $-\dfrac{2}{19}$

(iii) $-\dfrac{5}{19}$ and $-\dfrac{4}{19}$ (iv) $-\dfrac{9}{38}$ and $-\dfrac{4}{19}$.

(c) Let $ds^2 = -\left(dx^1\right)^2 - \left(dx^2\right)^2 - \left(dx^3\right)^2 + c^4\left(dx^4\right)^2$ be a metric defined on the Riemannian space V_4. Then the angle between two contravariant vectors (1, 0, 0, 1/c) and (–1, –1, 0, 1/c) is

(i) $\theta = \cos^{-1}\left(\dfrac{c^2+1}{\sqrt{c^2-1}\sqrt{c^2-2}}\right)$ (ii) $\theta = \cos^{-1}\left(\dfrac{c^2-1}{\sqrt{c^2+1}\sqrt{c^2-2}}\right)$

(iii) $\theta = \cos^{-1}\left(\dfrac{c^2+2}{\sqrt{c^2-1}\sqrt{c^2-2}}\right)$ (iv) $\theta = \cos^{-1}\left(\dfrac{c^2+1}{\sqrt{c^2-1}\sqrt{c^2+2}}\right)$.

(d) The intrinsic derivative of the fundamental tensor g_{ij} is

(i) 1 (ii) 0

(iii) -1 (iv) 2.

(e) The necessary and sufficient condition for a given curve to be a helix is that the ratio of the curvature to the torsion is always

(i) $(+)ve$ (ii) 0

(iii) $(-)ve$ (iv) Constant.

(f) If A_i is a covariant vector, then $\dfrac{\partial A_i}{\partial x^j}$ is

(i) a (0, 2) tensor (ii) a (2, 0) tensor

(iii) an (1, 1) tensor (iv) not a tensor.

(g) If g_{ij} is the fundamental metric tensor of type (0, 2) in a Riemannian space V_n. If A^i and B^i are two non-null contravariant vectors such that $g_{ij}u^iu^j = g_{ij}v^iv^j$ where $u^i = A^i + B^i$ and $v^i = A^i - B^i$, then

(i) A^i and B^i are parallel (ii) A^i and B^i are orthogonal

(iii) A^i and B^i are equal (iv) $g_{ij}\, A^i\, B^j = 1$.

(h) If A_{ij} is a symmetric tensor such that $A_{ij,k} = A_{ik,j}$ ($A_{ij,k}$ denotes the covariant derivatives of A_{ij} with respect to x^k), then $A_{ij,k}$ is

(i) a symmetric tensor of type (0, 3) (ii) a symmetric tensor of type (1, 2)

(iii) a symmetric tensor of type (2, 1) (iv) a skew-symmetric tensor of type (1, 2).

(i) The surface is developable if and only if

(i) $LN - M^2 > 0$ (ii) $LN - M^2 < 0$

(iii) $LN - M^2 = 0$ (iv) $LN - M^2$ is undefined.

(j) A surface M is a minimal surface if

(i) $k_1k_2 = 0$ (ii) $k_1 + k_2 = 0$

(iii) $K = 0$ (iv) $k_1 = k_2$.

where k_1, k_2 are principal curvatures and K is the Gaussian curvature.

Unit - 1

Answer ***any one*** question. 5×1

2. Prove that the components of a tensor of type (0, 2) can be uniquely expressed as the sum of a symmetric tensor and a skew symmetric tensor of the same type.

3. Find the Christoffel symbols $\left\{ {2 \atop 1\ 2} \right\}$ and $\left\{ {2 \atop 2\ 3} \right\}$ in a 3-dimensional Riemannian space in which the line element is given by $ds^2 = \left(dx^1\right)^2 + \left(x^1\right)^2 \left(dx^2\right)^2 + \left(x^1 \sin x^2\right)^2 \left(dx^3\right)^2$.

Unit - 2

Answer ***any four*** questions. 5×4

4. Prove that for an Einstein space V_n $(n > 2)$ the scalar curvature is constant.

5. Find the curvature and torsion of the curve $\vec{r} = \left(\tan^{-1} s\right)\hat{i} + \frac{1}{\sqrt{2}}\log\left(s^2+1\right)\hat{j} + \left(s - \tan^{-1} s\right)\hat{k}$.

6. Prove that necessary and sufficient condition for a vector field A to be parallel along the curve $\sigma : x^i = x^i(t),\ t_1 \le t \le t_2,\ i = 1, 2, 3$ is that

$$\frac{dA^i}{dt} + \left\{ {i \atop \alpha\,\beta} \right\} A^\alpha \frac{dx^\beta}{dt} = 0$$

where A^i, i = 1, 2, 3 are the components of A.

7. Find curvature and torsion at the point P of the curve σ, defined in cylindrical coordinates by equations

$$x^1 = a,\ x^2 = \theta(s),\ x^3 = 0$$

where the line element is given by $ds^2 = \left(dx^1\right)^2 + \left(x^1\right)^2\left(dx^2\right)^2 + \left(dx^3\right)^2$.

8. Prove that there are points on the cubic $x = at^3 + b$, $y = 3ct^2 + 3dt$, $z = 3et + f$ such that the osculating plane passes through the origin and that the points lie on the plane $3cex + afy = 0$.

9. If θ is the angle between the parametric curves, then show that $\cos\theta = \frac{a_{12}}{\sqrt{a_{11}a_{22}}}$.

10. Using the relation $K = \frac{b}{a}$, where $a = \det\left(a_{\alpha\beta}\right)$, $b = \det\left(b_{\alpha\beta}\right)$, derive the relation $K = \det\left(b^\alpha_\beta\right)$.

Unit - 3

Answer ***any four*** questions. 5×4

11. Find the differential equations of the geodesic for the metric $ds^2 = (du)^2 + (\sin u)^2(dv)^2$.

12. Find the Gaussian curvature for a surface with metric $ds^2 = a^2 \sin^2 u^1 \left(du^2\right)^2 + a^2\left(du^1\right)^2$.

13. Show that $x^1 = f_1\left(u^1\right), x^2 = f_2\left(u^1\right), x^3 = u^3$ is developable, where f_1, f_2 are differentiable functions.

14. Prove that along a line of curvature on a surface $\frac{\delta\xi^r}{\delta s} + \kappa_p \frac{dx^r}{ds} = 0$. Is the converse true? Justify your answer.

15. Find torsion of a geodesic in terms of principal curvature.

16. Using the Gauss–Bonnet theorem, prove that the Gaussian curvature is identically zero on a surface S if at any point P on S there are two families of geodesic curves in neighbourhood of P intersecting at a constant angle.

17. Prove that geodesic curvatures of u^1-curves and u^2-curves are respectively

$$\sigma_1 = \sqrt{\frac{a}{\left(a_{11}\right)^3}}\begin{Bmatrix} 2 \\ 1\ 1 \end{Bmatrix} \text{ and } \sigma_2 = -\sqrt{\frac{a}{\left(a_{22}\right)^3}}\begin{Bmatrix} 1 \\ 2\ 2 \end{Bmatrix}$$

where $a = \det\left(a_{\alpha\beta}\right)$.
