2021

MATHEMATICS — HONOURS

Sixth Paper

(Module - XI)

Full Marks : 50

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

Symbols and notations have their usual meanings.

Group - A
[Vector Calculus - II]
(Marks : 10)

1. Answer ***any two*** questions :

(a) Find the constants a, b, c so that the vector $\vec{V} = (x+2y+az)\hat{i} + (bx-3y-z)\hat{j} + (4x+cy+2z)\hat{k}$ is irrotational. Find a scalar function φ for which $\vec{V} = \vec{\nabla}\varphi$. 2+3

(b) Evaluate by Stoke's theorem $\oint_{\Gamma}(\sin z\, dx - \cos x\, dy + \sin y\, dz)$, where Γ is the boundary of the rectangle $0 \le x \le \pi, 0 \le y \le 1, z = 3$. 5

(c) Show that the circulation of the vector field $\vec{\alpha} = y^2\hat{i} + z^2\hat{j}$ around the contour $C : x^2 + y^2 = 1, y + z = 1$ is zero. 5

(d) Verify *Divergence Theorem* for $\boldsymbol{F} = 4xi - 2y^2j + z^2k$ taken over the region bounded by $x^2 + y^2 = 4$, $z = 0$ and $z = 3$. 5

Group - B
[Analytical Statics - II]
(Marks : 20)

Answer ***question no.*** **2** and ***any two*** questions from the rest.

2. (a) The thickness of a thin circular homogeneous plate at any point is proportional to the distance of the point from a tangent to its perimeter. Find the position of its centre of gravity, taking $2r$ to be its diameter. 6

Or,

(b) A uniform cubical box of edge a is placed on the top of a fixed sphere, the centre of the face of the cube in contact with the highest point of the sphere. Show that the least radius of the sphere for which the equilibrium will be stable is $\frac{a}{2}$. 6

3. A regular hexagon $ABCDEF$ consists of six equal rods each of weight W and are freely jointed together. The hexagon rests in a vertical plane and AB is in contact with a horizontal table; if C and F be connected by a light string, prove that the tension in the string is $W\sqrt{3}$. 7

4. A body rests in equilibrium on another fixed body, there being enough friction to prevent sliding. The portion of two bodies in contact are spherical and of radii r and R respectively and the line joining their centres in position of equilibrium is vertical. Show that the equilibrium is stable provided.

$$\frac{1}{h} > \frac{1}{r} + \frac{1}{R},$$

where h is the height of the C.G. of the body in position of equilibrium above the point of contact. 7

5. Define Poinsot's central axis. Let X, Y, Z and L, M, N denote respectively the algebraic sum of the components of a system of forces acting on a rigid body and their moments with respect to Cartesian axes Ox, Oy, Oz passing through any base point O. Show that the quantities $X^2 + Y^2 + Z^2$ and $LX + MY + NZ$ are invariants whatever be the base and the axes chosen to reduce the system. 2+5

6. OA, OB, OC are three co-terminus edges of a cube and AA', BB', CC', OO' are diagonals. The forces X, Y, Z and R act along BC', CA', AB' and OO' respectively. Show that the forces are equivalent to a single resultant if

$$(YZ + ZX + XY)\sqrt{3} + (X + Y + Z)R = 0$$ 7

Group - C

[Analytical Dynamics of a Particle - II]

(Marks : 20)

Answer ***question no.*** **7** and ***any two*** questions from the rest.

7. (a) For a satellite moving in a stable elliptic orbit, it is found that the minimum distance from the centre of the Earth is $4a$, where a is the radius of the Earth, and the period of one complete revolution is T. Show that eccentricity e of the orbit is given by

$$e = 1 - 4a\left(\frac{4\pi^2}{ga^2T^2}\right)^{\frac{1}{3}},$$

where g = acceleration due to gravity on the Earth's surface. 4

Or,

(b) A particle moves with a central acceleration $\mu\left(r+\dfrac{a^4}{r^3}\right)$ being projected from an apse at a distance a with velocity $2\sqrt{\mu a}$; show that it describes the curve $r^2\left(2+\cos\sqrt{3}\theta\right)=3a^2$. 4

8. Two particles of masses M and m are connected by a light string, which passes through a small hole in the table. The mass m hangs vertically, and M describes a curve on the table, which is very nearly a circle whose centre is the hole. Show that the apsidal angle of the orbit of M is $\pi\sqrt{\dfrac{M+m}{3M}}$. 8

9. If the period of a planet be 365 days and eccentricity e is $\dfrac{1}{60}$, show that the time of describing the two halves of the orbit bounded by the latus rectum passing through the centre of force, are $\dfrac{365}{2}\left(1 \pm \dfrac{1}{15\pi}\right)$, very nearly. 8

10. Classify the equilibrium point for the following linear dynamical system :

$$\dot{x}=x+3y,\ \dot{y}=x-y\ .$$

Also sketch the phase diagram. 6+2

11. If a particle of mass m be acted on by equal constant forces mf in the direction of the tangent and normal to its path, and if the resistance be mk^2v^2f, prove that the intrinsic equation of the path is $e^{2k^2fs}-1=k^2u^2\left(e^{2\psi}-1\right)$, where u is the velocity of projection of the particle. 8