

2021

## MATHEMATICS — HONOURS

Paper : CC-13

(Metric Space and Complex Analysis)

Full Marks : 65

*The figures in the margin indicate full marks.**Candidates are required to give their answers in their own words as far as practicable.*[ $\mathbb{N}$ ,  $\mathbb{R}$ ,  $\mathbb{Q}$ ,  $\mathbb{C}$  denote the set of all natural, real, rational and complex numbers respectively.]

(Notations and symbols have their usual meanings).

1. Answer **all** the following multiple choice questions. For each question **1** mark for choosing correct option and **1** mark for justification. 2×10

(a) Let  $(X, d)$  be a metric space and  $A, B \subseteq X$ . Choose the statement which is not true.

(i)  $\overline{(A \cup B)} = \overline{A} \cup \overline{B}$

(ii)  $A^\circ \cup B^\circ \subseteq (A \cup B)^\circ$

(iii)  $\partial(A \cap B) = \partial A \cap \partial B$

(iv)  $d(A, B) = d(\overline{A}, \overline{B})$ .

[  $\partial A$  denotes boundary of  $A$ ,  $d(A, B)$  denotes the distance between  $A, B$ . ](b) Two metrics  $d$  and  $d^*$  are defined on  $\mathbb{R}^2$  as follows :For all  $\hat{x} = (x_1, y_1), \hat{y} = (x_2, y_2) \in \mathbb{R}^2$ 

$$d(\hat{x}, \hat{y}) = \left[ (x_1 - x_2)^2 + (y_1 - y_2)^2 \right]^{1/2}$$

$$d^*(\hat{x}, \hat{y}) = \max\{|x_1 - x_2|, |y_1 - y_2|\}.$$

Then which of the following is true?

(i)  $d = d^*$

(ii)  $d$  and  $d^*$  are equivalent(iii)  $d$  and  $d^*$  are not equivalent(iv)  $(\mathbb{R}^2, d^*)$  is a submetric space of  $(\mathbb{R}^2, d)$ .(c) Let  $A = \{(x, y) : x, y \in \mathbb{R}, x \notin \mathbb{Q} \text{ or } y \notin \mathbb{Q}\}$ . Then which of the following is true with usual metric on  $\mathbb{R}^2$ ?(i)  $A$  is open but not compact in  $\mathbb{R}^2$ (ii)  $A$  is not open but compact in  $\mathbb{R}^2$ (iii)  $A$  is neither open nor compact in  $\mathbb{R}^2$ (iv)  $A$  is both open and compact in  $\mathbb{R}^2$ .

Please Turn Over





6. (a) Prove that a metric space  $(X, d)$  having the property that every continuous map  $f: X \rightarrow X$  has a fixed point, is connected.
- (b) Let  $(X, d)$  be a complete metric space and  $T: X \rightarrow X$  be a contraction on  $X$ . Then for  $x \in X$ , show that the sequence  $\{T^n x\}$  is a convergent sequence. 2+3
7. (a) Prove that the space  $\mathbb{Q}$  of rational numbers with subspace metric of the usual metric of  $\mathbb{R}$  is not connected.
- (b) Prove that  $(a, b]$  is connected with usual metric of  $\mathbb{R}$ . 2+3
8. Let  $(X, d_1)$  and  $(Y, d_2)$  be two metric spaces and  $f: (X, d_1) \rightarrow (Y, d_2)$  be uniformly continuous. Show that if  $\{x_n\}$  is a Cauchy sequence in  $(X, d_1)$  then so is  $\{f(x_n)\}$  in  $(Y, d_2)$ . Is it true if  $f$  is only continuous? Justify. 3+2
9. Let  $(X, d)$  be a complete metric space and let  $f: X \rightarrow X$  be a contraction mapping with Lipschitz constant  $t$  ( $0 < t < 1$ ). If  $x_0 \in X$  is unique fixed point of  $f$ , show that

$$d(x, x_0) \leq \frac{1}{1-t} d(x, f(x)), \forall x \in X \quad 5$$

### Unit - 2

#### (Complex Analysis)

Answer *any four* questions.

10. (a) Let  $z_1$  and  $z_2$  be the images in the complex plane of two diametrically opposite points on the Riemann sphere under stereographic projection. Then show that
- $$z_1 \bar{z}_2 = -1.$$
- (b) Prove or disprove : The image of the circle  $|z| = r$  ( $r \neq 1$ ) under the transformation  $w = z + \frac{1}{z}$  is an ellipse. 2+3
11. Let  $f: \mathbb{C} \rightarrow \mathbb{C}$  be defined by  $f(z) = \frac{\bar{z}^2}{z}$  for  $z \neq 0$  and  $f(0) = 0$ . Show that the Cauchy Riemann equations are satisfied at  $z = 0$ , but the derivative of  $f$  fails to exist there. 5
12. (a) If  $f$  is analytic function of  $z = x + iy \in \mathbb{C}$  and  $\bar{z} = x - iy$ , then show that  $\frac{\partial f}{\partial \bar{z}} = 0$ .
- (b) Let  $f$  be analytic on a region  $G$ . If  $f$  assumes only real values on  $G$ , then show that  $f$  is a constant function. 3+2

13. Let  $f$  be an analytic function in a region  $G$ . Show that  $Re(f)$  satisfies the following equation :

$$\frac{\partial^2 \Psi}{\partial r^2} + \frac{1}{r} \frac{\partial \Psi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \Psi}{\partial \theta^2} = 0. \quad 5$$

14. Let  $f(z) = \sum_{n=0}^{\infty} a_n z^n$  be a power series with radius of convergence  $R > 0$ . Show that  $f$  is differentiable

on  $|Z| < R$ . Show that  $\frac{df}{dz} = \sum_{n=1}^{\infty} n a_n z^{n-1}$  and it has radius of convergence  $R$ . 5

15. (a) Find the radius of convergence of the power series  $\sum_{n=0}^{\infty} a_n z^n$ , where  $a_n = \frac{e^{in\pi}}{n}$ .

(b) Let  $f(z) = \bar{z}$  and  $\gamma$  is the semicircle from 1 to  $-1$  passing through  $i$ . Evaluate  $\int_{\gamma} f(z) dz$ . 2+3

16. (a) Evaluate :  $\int_{|z|=2} \frac{e^z + z^2}{z-1} dz$

(b) Show that  $\left| \int_{\gamma} \frac{dz}{z^2 + 4} \right| \leq \frac{\pi R}{(R^2 - 4)}$ , where  $\gamma(t) = Re^{it}$  for  $0 \leq t \leq \pi$  and  $R > 2$ . 3+2

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