2020

PHYSICS — HONOURS

Paper : CC-1

(Mathematical Physics I)

Full Marks : 50

The figures in the margin indicate full marks.
Candidates are required to give their answers in their own words as far as practicable.

Answer ***question no.*** **1** and ***any four*** questions from the rest.

1. Answer ***any five*** questions : 2×5

(a) Evaluate $\lim_{x\to 1}\frac{x^{10}-1}{x^5-1}$

(b) State the order and degree of the differential equation

$$\frac{d^3y}{dx^3}+\left(\frac{dy}{dx}\right)^2+x^3y=0$$

(c) Check whether the three vectors $\hat{i}, \hat{i}+\hat{j}, \hat{i}+\hat{j}+\hat{k}$ are linearly independent.

(d) Check whether $dw = 2xy\,dx + x^2dy$ is an exact differential.

(e) If $\vec{A}$ is a constant vector, find $\vec{\nabla}\left(\vec{A}.\vec{r}\right)$.

(f) Show that any square matrix can be written as the sum of a symmetric and an anti-symmetric matrix.

(g) A 2×2 matrix A satisfies the equation $(A-2I)^2 = O$, where I is the 2×2 identity matrix. Find the trace of A.

2. (a) Sketch the function e^x, e^{-x} and $e^{-|x|}$ for $-1 \le x \le 1$. Explain whether the function $e^{-|x|}$ is differentiable at $x = 0$.

(b) Find the Taylor series expansion of sin x about $x = \pi$, giving the first two non-zero terms.

(c) Show that the functions x, x^2 and x^3 are linearly independent. (3+2)+3+2

3. (a) Solve the equation

$$y'' + 6y' + 8y = 0$$

subject to the condition $y = 1, y' = 0$ at $x = 0$ where $y' \equiv \frac{dy}{dx}$ and $y'' \equiv \frac{d^2y}{dx^2}$.

(b) Prove that if the Wronskian of the functions $f_1(x), f_2(x)$ and $f_3(x)$ is not identically zero, then the functions are linearly independent.

(c) Given $x = r\cos\theta, y = r\sin\theta$

calculate $\left(\frac{\partial\theta}{\partial x}\right)_y, \left(\frac{\partial x}{\partial\theta}\right)_y$ and $\left(\frac{\partial x}{\partial\theta}\right)_r$. 4+3+3

4. (a) Considering two position vectors in three dimensions, show that their scalar product remains invariant under the rotation of co-ordinate system about z-axis.

(b) Suppose that the temperature T at any point (x, y, z) is given by

$T(x, y, z) = x^2 - y^2 + yz + 373.$

In which direction is the temperature increasing most rapidly at $(-1, 2, 3)$? What is the maximum rate of change of temperature at that point?

(c) If S is any closed surface enclosing a volume V and $\vec{A} = ax\hat{i} + by\hat{j} + cz\hat{k}$, , prove that $\oiint_S \vec{A}.d\vec{S} = (a+b+c)V$. 4+(2+1)+3

5. (a) Using Gauss' divergence theorem, show that

$$\iiint_V \left(\phi\nabla^2\psi - \psi\nabla^2\phi\right)dV = \oiint_S \left(\phi\vec{\nabla}\psi - \psi\vec{\nabla}\phi\right).d\vec{S}$$

where $\phi(x, y, z)$ and $\psi(x, y, z)$ are two scalar functions and the surface integral is over the surface S enclosing the volume V.

(b) Prove that $\vec{\nabla}\times\left(\phi\vec{V}\right) = \left(\vec{\nabla}\phi\right)\times\vec{V} + \phi\left(\vec{\nabla}\times\vec{V}\right)$ for a scalar field $\phi(x, y, z)$ and a vector field $\vec{V}(x, y, z)$.

Now take $\vec{V}$ to be a non-zero constant vector field $\vec{C}$ and use Stokes′ theorem to prove that $\oint_C \phi d\vec{r} = \iint_S d\vec{S}\times\vec{\nabla}\phi$, where the closed curve C is the boundary of the surface S.

(c) Let $\hat{\rho}$ and $\hat{\phi}$ be the unit vectors in plane polar coordinates. If

$$\begin{pmatrix}\hat{\rho}\\ \hat{\phi}\end{pmatrix} = R\begin{pmatrix}\hat{i}\\ \hat{j}\end{pmatrix},$$

find the matrix R. 3+(2+3)+2

6. (a) Find the eigenvalues and the normalized eigenvectors of the matrix

$$M = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

Write down the matrix U such that $U^{-1}MU$ is diagonal.

(b) Consider $AB - BA = iC$. If A and B are hermitian matrices, show that the matrix C is also hermitian. Here $i = \sqrt{-1}$.

(c) Show that an eigenvector of a matrix A with eigenvalue λ is also an eigenvector of the matrix A^3 with eigenvalue λ^3. (2+2+2)+2+2

7. (a) If C is an orthogonal matrix and M is a symmetric matrix, show that $C^{-1}MC$ is symmetric.

(b) Show that the eigenvectors of a hermitian matrix with different eigenvalues are orthogonal.

(c) Solve the system of equations

$$\frac{dx}{dt} = 2x + 3y$$
$$\frac{dy}{dt} = 3x - 6y.$$

3+3+4